Problem Sheet 5 for Supervision in Week 12

1. ★ Use the table below to sieve the integers up to 200 for primes. Thus calculate $\pi(200)$.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160
161	162	163	164	165	166	167	168	169	170
171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190
191	192	193	194	195	196	197	198	199	200

What is the smallest composite number that has no prime factor in the table?

Are the following numbers prime or composite?

(i) 44517 (ii) 44503 (iii) 44519.

2.
   i) Find the four smallest sets of prime triplets of the form $p, p+4$ and $p+6$.
   ii) Why are there no prime triplets of the form p, $p+2$ and $p+4$, other than $(3,5,7)$?
   (Hint, look at the p modulo 3.)

3. ★ Use Fermat's Little Theorem and Euler's Theorem to

   i) show that $5555^{2222} + 2222^{5555}$ is divisible by 7,

   ii) show that $5555^{2222} + 2222^{5555}$ is divisible by 3 but not by 9,

   iii) find the last two digits in the decimal expansion of $3333^{7777} + 7777^{3333}$.

4. i) Calculate $7^5 \bmod 13$ and $7^7 \bmod 13$.

   ii) Using Fermat's Little Theorem, along with part i, solve $6x \equiv 5 \bmod 13$. (Do not use Euclid's algorithm.)

5. Using the method of successive squaring calculate $2^{90} \bmod 91$. Hence show that 91 is not prime.

6. Show that Euler's phi function evaluated at prime powers satisfies $\phi\left(p^k\right) = p^{k-1}\left(p-1\right)$.

   Hint. Instead of counting the set of integers coprime to p^k count the complement of this set.

7. ★ Let

$$\begin{aligned}
\sigma_1 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 1 & 3 & 6 \end{pmatrix} \in S_6, \\
\sigma_2 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 6 & 2 & 5 \end{pmatrix} \in S_6, \\
\sigma_3 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 1 & 4 & 5 \end{pmatrix} \in S_6, \\
\tau_1 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 1 & 9 & 8 & 6 & 4 & 2 & 5 & 3 \end{pmatrix} \in S_9, \\
\tau_2 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 7 & 5 & 9 & 8 & 4 & 2 & 3 & 6 \end{pmatrix} \in S_9.
\end{aligned}$$

   (a) Calculate $\sigma_1\sigma_2,\ \sigma_2\sigma_3,\ \sigma_3\sigma_1,\ \sigma_1^2,\ \sigma_3^3,\ \sigma_1\sigma_2\sigma_1,\ \tau_1\tau_2,\ \tau_2\tau_1\tau_2$ and τ_1^4.

   (b) Find the inverses of $\sigma_1,\ \sigma_2,\ \sigma_1\sigma_2,\ \sigma_3,\ \tau_1$ and τ_2.

   (c) Verify that $\left(\sigma_1\sigma_2\right)^{-1} = \sigma_2^{-1}\sigma_1^{-1}$.